Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 51 (2001), No. 2, 225--229

Persistent URL: http://dml.cz/dmlcz/127643

SIGNED TOTAL DOMINATION NUMBER OF A GRAPH

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(Received September 25, 1995)

Abstract. The signed total domination number of a graph is a certain variant of the domination number. If v is a vertex of a graph G, then $N(v)$ is its oper neighbourhood, i.e. the set of all vertices adjacent to v in G. A mapping $f\colon V(G) \to \{-1, 1\}$, where $V(G)$ is the vertex set of G, is called a signed total dominating function (STDF) on G, if $\sum\limits_{x \in N(v)} f(x) \geqslant 1$ for each $v \in V(G)$. The minimum of values $\sum\limits_{x \in V(G)} f(x)$, taken over all STDF's of G, is called the signed total domination number of G and denoted by $\gamma_{\rm st}(G)$. A theorem stating lower bounds for $\gamma_{\rm st}(G)$ is stated for the case of regular graphs. The values of this number are found for complete graphs, circuits, complete bipartite graphs and graphs on n-side prisms. At the end it is proved that $\gamma_{\rm st}(G)$ is not bounded from below in general.

Keywords: signed total dominating function, signed total domination number, regular graph, circuit, complete graph, complete bipartite graph, Cartesian product of graphs

MSC 2000: 05C69, 05C35

In this paper we study the signed total domination number of a graph. This concept is obtained by a modification of the total domination number; this modification is analogous to that which leads from the domination number to the signed domination number.

First we define necessary concepts and notation. We consider finite undirected graphs without loops and multiple edges. The vertex set of a graph G is denoted by $V(G)$. If $v \in V(G)$, then the open neighbourhood $N(v)$ of v in G is the set of all vertices which are adjacent to v in G. Further, the closed neighbourhood of v in G is defined as $N[v] = N(v) \cup \{v\}$.

Let f be a mapping of $V(G)$ into some set of numbers, let $S \subseteq V(G)$. Then we denote $f(S) = \sum\limits_{x \in S} f(x)$. Further, the weight of f is $w(f) = f\bigl(V(G)\bigr) = \sum\limits_{x \in V(G)} f(x)$.

A set $D \subseteq V(G)$ is called dominating (or total dominating) in G, if $D \cap N[v] \neq \emptyset$ (or $D \cap N(v) \neq \emptyset$, respectively) for every vertex $v \in V(G)$. The minimum number of vertices of a dominating set (or of a total dominating set) in G is called the domination number $\gamma(G)$ (or the total domination number $\gamma_{\rm t}(G)$, respectively) of G.

It is possible to speak about the characteristic functions of dominating and total dominating sets. A characteristic function f of a set $D \subseteq V(G)$ is a mapping $F\colon V(G) \to \{0,1\}$ such that $f(v) = 1$ if and only if $v \in D$; otherwise $f(v) = 0$. Evidently, D is dominating in G if and only if $f(N[v]) \geqslant 1$ for each $v \in V(G)$.

The dominating number of G is the minimum of $w(f)$ taken over all such functions. Similarly, D is total dominating in G if and only if $f\bigl(N(v)\bigr) \geqslant 1$ for each $v \in V(G)$ and the total domination number of G can be defined as the minimum of $w(f)$ taken over all such functions.

A modification of the definition of the domination number $\gamma(G)$ (this concept was described e.g. in [2]) led in [1] to the definition of the signed domination number $\gamma_{\rm s}(G)$. Instead of a function $f\colon V(G) \to \{0,1\}$ a function $f\colon V(G) \to \{-1,1\}$ was considered. We have the following defintions.

A function $f\colon V(G) \to \{-1,1\}$ is called a signed dominating function (shortly SDF) of G, if $f(N[v]) \geqslant 1$ for each $v \in V(G)$. The minimum of $w(f) = f\bigl(V(G)\bigr) = \sum\limits_{x \in V(G)} f(x)$, taken over all SDF's of G, is the signed domination number $\gamma_{\rm s}(G)$ of G.

Quite analogously also the other definition may be modified.

A function $f\colon V(G) \to \{-1,1\}$ is called a signed total dominating function (shortly STDF) of G, if $f\bigl(N(v)\bigr) \geqslant 1$ for each $v \in V(G)$. The minimum of $w(f) = f\bigl(V(G)\bigr) = \sum\limits_{x \in V(G)} f(x)$, taken over all STDF's of G, is the signed total domination number $\gamma_{\rm st}(G)$ of G.

We will study this concept. We start by a lemma.

Lemma. *Let $f\colon V(G) \to \{-1,1\}$ and $S \subseteq VG$. Then $f(S) \equiv |S| \pmod 2$.*

Proof. Let $S^+ = \{x \in S \mid f(x) = 1\}$, $S' = \{x \in S \mid f(x) = -1\}$. Then $|S| = |S^+| + |S^-|_{\rm st} f(S) = |S^+| - |S^-|$ and $|S| - f(S) = 2|S^-|$, which implies the assertion. $\square$

We shall prove a theorem concerning regular graphs.

Theorem 1. *Let G be a regular graph of degree r. If r is odd, then $\gamma_{\rm st}(G) \geqslant n/r$; if r is even, then $\gamma_{\rm st}(G) \geqslant 2n/r$.*

Proof. Let f be a STDF od G such that $w(f) = \gamma_{\rm st}(G)$. Let $V^+ = \{v \in V(G) \mid f(v) = 1\}$, $V^- = \{v \in V(G) \mid f(v) = -1\}$, let E_0 be the set of all edges joining a vertex of V^+ with a vertex of V^- in G. Let $u \in V^+$ and let u be adjacent to exactly

s vertices of V^-. Then u is adjacent to $r-s$ vertices of V^+ and $f\bigl(N(u)\bigr) = r-2s \geqslant 1$, which implies $s \leqslant \frac{1}{2}(r-1)$. Therefore u is adjacent to at most $\frac{1}{2}(r-1)$ vertices of V^-. Now let $v \in V^-$ and let v be adjacent to exactly t vertices of V^+. Then v is adjacent to $r-t$ vertices of V^- and $f\bigl(N(v)\bigr) = 2t - r \geqslant 1$, which implies $t \geqslant \frac{1}{2}(r+1)$. If $n^+ = |V^+|$, $n^- = |V^-|$, then $|E_0 \leqslant \frac{1}{2}n^+(r-1)$ and simultaneously $|E_0| \geqslant \frac{1}{2}n(r+1)$. This implies $\frac{1}{2}n^-(r+1) \leqslant \frac{1}{2}n^+(r-1)$ and further $n^+ + n^- \leqslant (n^+ - n^-)r$, which is $n \leqslant \gamma_{\rm st}(G)r$ and $\gamma_{\rm st}(G) \geqslant n/r$. If, moreover, r is even, then $r - 2s \geqslant 2$, $2t - r \geqslant 2$ and this yields $\gamma_{\rm st}(G) \geqslant 2n/r\}$. $\square$

This theorem enables us to determine $\gamma_{\rm st}(G)$ for certain classes of regular graphs.

Proposition 1. *For a complete graph K_n we have $\gamma_{\rm st}(K_n) = 1$ for n odd and $\gamma_{\rm st}(K_n) = 2$ for n even.*

P r o o f. The number $\gamma_{\rm st}(K_n)$ cannot be less than the value in this result. This value is attained by a STDF f such that $f(v) = 1$ for $\frac{1}{2}(n+1)$ vertices v in the case of n odd and $f(v) = 0$ for $\frac{1}{2}n + 1$ vertices v in the case of n even. $\square$

Proposition 2. *For a circuit C_n of length n we have $\gamma_{\rm st}(C_n) = n$.*

P r o o f. Here no other STDF exists than the constant equal to 1. $\square$

Proposition 3. *For the Cartesian product $C_n \times K_2$ (graph of the n-side prism) we have $\gamma_{\rm st}(C_n \times K_2) = \lceil 2n/3 \rceil$.*

P r o o f. We obtain the necessary STDF in such a way that we choose the maximum number of vertices on one copy of C_n such that the distance between arbitrary two of them is at least 3, add to them the vertices of the other copy which are adjacent to them, and to all of these vertices we assign the value -1, while to all others we assign the value 1. $\square$

For $n = 6$ we see the situation in Fig. 1. This case shows us another interesting fact. It is well-known that each total dominating set in a graph G is also a dominating set in G. This implies $\gamma(G) \leqslant \gamma_{\rm t}(G)$. But it is not true that each STDF in G is also a SDF in G. The function in Fig. 1 is an example. Moreover, in fact, $4 = \gamma_{\rm st}(C_n \times K_2) < \gamma_{\rm st}(C_n \times K_2) = 8$. The corresponding SDF is in Fig. 2. Note that any two distinct vertices with value -1 in a SDF must have the distance at least 3, but those in a STDF may be adjacent.

Proposition 4. *For a complete bipartite graph $K_{m,n}$ we have $\gamma_{\rm st}(K_{m,n}) = 2$ in the case of both m, n odd, $\gamma_{\rm st}(K_{m,n}) = 3$ in the case when m, n are different parity, and $\gamma_{\rm st}(K_{m,n}) = 4$ in the case of both m, n even.*

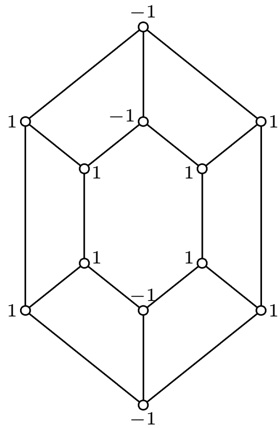

Fig. 1.

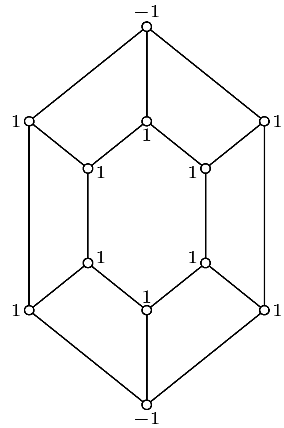

Fig. 2.

P r o o f. Let the bipartition classes of $K_{m,n}$ be V_1, V_2 and $|V_1| = m$, $|V_2| = n$. For $v \in V_1$ we have $N(v) = V_2$ and for $v \in V_2$ we have $N(v) = V_1$. Therefore if f is a STDF such that $w(f) = \gamma_{\rm st}(K_{m,n})$ then $f(V_1) \geqslant 1$, $f(V_2) \geqslant 1$. Moreover, $f(V_1) \geqslant 2$ for m even and $f(V_2) \geqslant 2$ for n even by Lemma. The equality in all the mentioned cases can be attained in a similar way as in the complete graph. Then $w(f) = f(V_1) + f(V_2)$ and this yields the result. $\square$

The next proposition shows that the signed total domination number is not well-defined for all graphs.

Proposition 5. *The signed total domination number $\gamma_{\rm st}(G)$ of a graph G is well-defined if and only if G has no isolated vertex.*

P r o o f. Let G contain an isolated vertex v. Then $N(v) = \emptyset$ and $f\big(N(v)\big) = 0$ for each $f\colon V(G) \to \{-1, 1\}$. Therefore no STDF exists in G. If G contains no isolated vertex, then $N(v) \neq \emptyset$ for each $v \in V(G)$. There exists at least one STDF of G, namely the function f such that $f(v) = 1$ for each $v \in V(G)$. Therefore it is meaningful to speak about the minimum weight of a STDF of G. $\square$

In the last proposition we show that in general $\gamma_{\rm st}(G)$ is not bounded from below.

Proposition 6. *Let $k \geqslant 4$ be an integer. Then there exists a graph G such that $|V(G)| = k(k+1)$ and $\gamma_{\rm st}(G) = k(3-k)$.*

P r o o f. Take a graph H isomorphic to the complete graph K_k. To each $v \in V(H)$ assign a star $S(v)$ with $k-2$ edges. Identify the central vertex of $S(v)$ with v for each $v \in V(H)$. Denote the resulting graph by G. Let $f\colon V(G) \to \{-1, 1\}$ be such that $f(v) = 1$ for $v \in V(H)$ and $f(v) = -1$ otherwise. For each $v \in V(G)$ we have $f\big(N(v)\big) = 1$. Therefore f is a STDF of G; its weight is $k(3-k)$.This weight is minimum, because for each $v \in V(H)$ and for each STDF g we must have $g(v) = 1$: namely $\{v\} = N(x)$ for each $x \in V\big(S(v)\big) - \{v\}$. Therefore $\gamma_{\rm st}(G) = k(3-k)$. $\square$

An example for $k = 5$ is in Fig. 3.

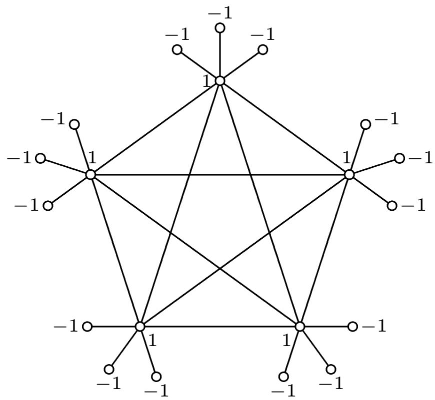

Fig. 3.

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